

Constructing Analytically Tractable Ensembles of Non-Stationary Covariances with an Application to Financial Data

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In complex systems, crucial parameters are often subject to unpredictable changes in time. Climate, biological evolution and networks provide numerous examples for such non-stationarities. In many cases, improved statistical models are urgently called for. In a general setting, we study systems of correlated quantities to which we refer as amplitudes. We are interested in the case of non-stationarity, *i.e.*, seemingly random covariances. We present a general method to derive the distribution of the covariances from the distribution of the amplitudes. To ensure analytical tractability, we construct a properly deformed Wishart ensemble of random matrices. We apply our method to financial returns where the wealth of data allows us to carry out statistically significant tests. The ensemble that we find is characterized by an algebraic distribution which improves the understanding of large events.

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I. INTRODUCTION

In electroencephalography (EEG) electrical currents are recorded at different positions on the scalp to measure the brain activity. The correlations between the time series of these currents strongly depend on the overall state of the brain. During an epileptic seizure, for example, the correlations are much stronger than in normal periods [1, 2]. This time dependence of the correlations is the kind of non-stationarity that we wish to address. Non-stationarities are also seen when wave packets travel through disordered systems. Even if the disorder is static, the correlations between the wave intensities measured at different positions versus time will change, when the direction or the composition of the wave packet is altered [3–5]. Finance provides another important example for this type of non-stationarity. The correlations between stock price time series change in time, just as the business relations between the firms and the traders' market expectations [6–11]. Similar non-stationarities exist in many complex systems, including velocity fluctuations in turbulent flows, heartbeat dynamics, series of waiting times, etc. [12–15].

A system showing non-stationary correlations may be interpreted as being out of equilibrium, implying that some of the key tools in statistical physics are not applicable. Yet, the challenges are similar to the one faced for equilibrium systems: Is there generic or universal behavior? — How can we identify it? — Can we set up statistical models for these non-stationarities? — In the context of finance, we recently put forward a random ma-

trix approach to tackle these issues [16]. We also successfully applied it in a study of credit risk and its impact on systemic stability [17]. In spite of the conceptual differences, random matrix theory [18, 19] formally has much in common with statistical mechanics. Observables are averaged over an ensemble; in statistical mechanics, it usually is the microcanonical, canonical or macrocanonical one, in random matrix models, it is an ensemble of those matrices which describe or characterize the system. In the context of the present discussion, random matrix models can be divided into two classes:

1. The ensemble is fictitious. It comes into play via an ergodicity argument only.
2. The ensemble really exists and can be identified in the system. The issue of ergodicity does not arise.

The vast majority of random matrix models in, *e.g.*, quantum chaos falls into class 1, for a review see Ref. [19]. One is interested in the spectral statistics of one individual system. Its Hamiltonian is viewed as a random matrix, whose dimension is eventually sent to infinity. Ergodicity holds in this limit, meaning that a smoothing energy average of an observable over one individual spectrum equals the average over an ensemble of random matrices. A noticeable exception are random matrix applications to quantum chromodynamics [20]. In lattice gauge theory, the quarks first propagate in frozen configurations of the gauge fields, before an average over the gauge fields, modeled by random matrices, is carried out as second step. This clearly belongs in class 2. The fluctuating gauge fields truly exist, the partition function involves an integral over them. Ergodicity reasoning is not evoked.

There are numerous applications of random matrix theory in finance [21–31] which address statistical properties of correlation matrices. Many of them also deal with non-Gaussian ensembles. To the best of our knowledge, all of these applications fall into class 1, because

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one is interested in the statistics of one individual correlation matrix, measured at one particular instant in time. In our study [16], we put forward a first application of random matrices in finance that belongs in class 2. Non-stationarity makes the covariances fluctuate and thereby creates an ensemble of covariance matrices which we approximated by a Gaussian Wishart ensemble of random matrices [32]. We derived how the multivariate distribution of dimensionless price changes, referred to as returns, acquires heavy tails due to the non-stationarity. Hence, we showed that the non-stationarities indeed have universal features.

Here, we have three goals: First, we present in Sec. II a statistically significant way to construct a proper and analytically tractable random matrix ensemble from the data. We emphasize that this is an important issue for random matrix models in the context of correlations. In contrast to quantum chaos, where universality holds on the scale of the mean level spacing, there is not such a local scale when studying statistical properties of correlation matrices. Thus, a Gaussian assumption is not always justified and it does matter what the ensemble looks like in reality. In particular, realistic ensembles considerably help to understand and model large events. Our construction is general and not tied to any specific system. Its merit lies in the fact that once the ensemble is known, it can be used to work out generic statistical properties of any observable depending on the correlation matrices, see Ref. [17] for an example. Second, we apply our approach to financial data in Sec. III. We identify an algebraic ensemble, which is quite relevant for risk estimation. Third, we discuss two issues arising in our general construction in Secs. IV and V, namely a certain conceptual caveat and yet a further extension, respectively. Conclusions are given in Sec. VI.

II. CONSTRUCTING A PROPER RANDOM MATRIX ENSEMBLE

After setting up the general problem in Sec. II A, we introduce the deformed Wishart ensemble and derive the corresponding amplitude distribution in Sec. II B. The determination of the deformation functions which characterize the ensemble and the amplitude distribution is discussed in Sec. II C. Here, we derive the approach for the general case, for sake of illustration, the reader is referred to Ref. [16] and Sec. III

A. Non-Stationary Covariances

Suppose we have measured in a system with randomness K amplitudes as time series $R_k(t)$, $k = 1, \dots, K$ over a long interval T_{tot} of time $t = 1, \dots, T_{\text{tot}}$. For examples, these amplitudes can be electric or magnetic fields at K different points in a disordered system, positions of K randomly moving particles or financial returns, *i.e.*

dimensionless price changes for K stocks. Importantly, we assume that there are correlations between the time series. In complex systems, one often encounters the situation that crucial system parameters, in particular the covariances or correlations, are seemingly random functions of time [33–35]. To be more precise, we consider a time window of length T that is much shorter than the total interval, $T \ll T_{\text{tot}}$. We now want to average over the subinterval $[t - T + 1, t]$ of length T whose position in the total interval is determined by the time t . Sample averages of a function $f(t)$ in this subinterval are then written as

$$\langle R_k \rangle_T(t) = \frac{1}{T} \sum_{t'=t-T+1}^t f(t'). \quad (1)$$

We are particularly interested in the covariances

$$\begin{aligned} \Sigma_{kl}(t) &= \langle R_k R_l \rangle_T(t) - \langle R_k \rangle_T(t) \langle R_l \rangle_T(t) \\ &= \langle r_k r_l \rangle_T(t), \end{aligned} \quad (2)$$

where we introduced the amplitudes normalized to zero mean value,

$$r_k(t) = R_k(t) - \langle R_k \rangle_T(t). \quad (3)$$

We keep in mind that the resulting $K \times K$ covariance matrix $\Sigma(t)$ is calculated from time series of length T . We now move this time window of length T through the data, the resulting covariances $\Sigma_{kl}(t)$ fluctuate. This non-stationarity has an important impact on other statistical observables. In the present study, we focus on the distribution of the amplitudes. We now consider a time interval T as short as possible such that the covariance matrix Σ_s in this time interval is in good approximation constant. We begin with addressing the case in which the distribution of the amplitudes is, for a given time t , well approximated by a multivariate Gaussian

$$g(r|\Sigma_s) = \frac{1}{\sqrt{\det(2\pi\Sigma_s)}} \exp\left(-\frac{1}{2}r^\dagger \Sigma_s^{-1} r\right) \quad (4)$$

with the K component vector $r = (r_1, \dots, r_K)$ and the $K \times K$ covariance matrix Σ_s . We suppress the argument t of r and use \dagger to indicate the transpose. We refer to $g(r|\Sigma_s)$ as *static amplitude distribution*. Due to the correlations, a Gaussian assumption for the static distribution is not as restrictive as it may seem. In the eigenbasis of Σ_s , the amplitudes only appear in linear combinations. Thus, for large K , the mechanisms that lead to the central limit theorem start working and drive the distributions towards Gaussians. Later on in Sec. V we will nevertheless relax the Gaussian assumption for the static amplitude distribution and look at more general functional forms.

B. Deformed Wishart Ensemble and its Amplitude Distribution

How does the non-stationarity affect the amplitude distribution when data from the total interval T_{tot} are

analyzed? — As in Ref. [16], we model this by random matrices. As the covariance matrix is different at each time t where it is analyzed, we replace the covariance matrix in the distribution (4) by the expression

$$\Sigma_s \longrightarrow \frac{1}{N} AA^\dagger, \quad (5)$$

where A is a real rectangular $K \times N$ random matrix without any symmetries. The right hand side of Eq. (5) has to have the form given to ensure that it can model a properly defined covariance matrix. This follows directly from the definition (2). Although K , the first dimension is fixed, the second one, N , is for the time being a free model parameter. It can be viewed as the length of the model time series. Further clarifications will follow. To obtain the amplitude distribution for the total interval, we average over the random matrices

$$\langle g \rangle(r|\Sigma, N) = \int d[A] \bar{w}(A|\Sigma, N) g\left(r \left| \frac{1}{N} AA^\dagger \right.\right), \quad (6)$$

where $d[A]$ is the volume element, *i.e.*, the product of all independent variables in A . Following Wishart [32, 36], the Gaussian distribution

$$w(A|\Sigma) = \frac{1}{\det^{N/2}(2\pi\Sigma)} \exp\left(-\frac{1}{2} \text{tr} A^\dagger \Sigma^{-1} A\right) \quad (7)$$

was assumed for the random matrices in Ref. [16]. It describes the Gaussian fluctuations of the model covariance matrices AA^\dagger/N about the given empirical covariance matrix Σ , which is evaluated over the total time interval of length T_{tot} . It should not be confused with the empirical covariance matrix $\Sigma(t)$ calculated in the subintervals $[t - T + 1, t]$. The crucial difference compared to Ref. [16] is a generalization of the ensemble (7). We introduce the deformed Wishart ensemble

$$\bar{w}(A|\Sigma, N) = \int_0^\infty d\eta f(\eta) w\left(A \left| \frac{N\Sigma}{\eta} \right.\right) \quad (8)$$

which is defined by the *ensemble deformation function* $f(\eta)$ with the properties

$$\int_0^\infty f(\eta) d\eta = 1 \quad \text{and} \quad f(\eta) \geq 0. \quad (9)$$

For later convenience, Σ on the right hand side of Eq. (8) is rescaled with N . The fluctuations of the model covariance matrices AA^\dagger/N deviate from Gaussian, but always about the empirical covariance matrix Σ . The meaning of the model parameter N now becomes clearer. It sets the variance for these fluctuations. The above rescaling only changes the functional dependencies, but not the rôle of N . We emphasize once more that Σ is evaluated over the total time interval. Similar deformations of random matrix ensembles but in a Hamiltonian, not Wishart setting were apparently first put forward in Refs. [37, 38].

After inserting the ansatz (8) into Eq. (6), we may use the result [16]

$$\int w(A|\Sigma) g\left(r \left| \frac{1}{N} AA^\dagger \right.\right) d[A] = \int_0^\infty \chi_N^2(z) g\left(r \left| \frac{z}{N} \Sigma \right.\right) dz, \quad (10)$$

which reformulates the whole random matrix average as a univariate average over the χ^2 distribution

$$\chi_N^2(z) = \frac{1}{2^{N/2} \Gamma(N/2)} z^{N/2-1} \exp\left(-\frac{z}{2}\right) \quad (11)$$

of N degrees of freedom. On the mathematical side, there are connections between formula (10) and the calculation of certain distributions in scattering theory [39, 40]. Using the result (10), the amplitude distribution reduces to the double integral

$$\langle g \rangle(r|\Sigma, N) = \int_0^\infty d\eta f(\eta) \int_0^\infty dz \chi_N^2(z) g\left(r \left| \frac{z}{\eta} \Sigma \right.\right). \quad (12)$$

Again, we point out the rescaling of Σ with N , *cf.* Eq. (10). It is useful to rewrite that as a single integral

$$\langle g \rangle(r|\Sigma, N) = \int_0^\infty p(x) g(r|x\Sigma) dx \quad (13)$$

by introducing the variable $x = z/\eta$ and its distribution

$$p(x) = \int_0^\infty d\eta f(\eta) \int_0^\infty dz \chi_N^2(z) \delta\left(x - \frac{z}{\eta}\right). \quad (14)$$

We refer to it as *amplitude distribution deformation function*. One easily obtains

$$p(x) = \frac{x^{N/2-1}}{2^{N/2} \Gamma(N/2)} \int_0^\infty d\eta f(\eta) \eta^{N/2} \exp\left(-\frac{\eta x}{2}\right), \quad (15)$$

which establishes the relation between the two deformation functions.

We notice that the ansatz (8) restricts the form of the deformed distribution $\bar{w}(A|\Sigma, N)$ to functions of $\text{tr} A^\dagger \Sigma^{-1} A$ only. Even though the inclusion of further terms such as $\text{tr}(A^\dagger \Sigma^{-1} A)^2$ is likely to improve the quality of the data fits, we stick to the ansatz (8). Its considerable advantage is the guaranteed but otherwise questionable analytical tractability as will be shown in the sequel. Moreover, further terms will also increase the number of deformation functions which will hamper their unambiguous determination.

C. Determination of the Deformation Functions

Apart from the deformation functions, the distributions $\bar{w}(A|N\Sigma)$ and $\langle g \rangle(r|\Sigma, N)$ depend on the usual covariance matrix Σ analyzed by sampling over the total

interval. We notice that the corresponding covariance matrix $\Sigma^{(d)}$ in the deformed ensemble slightly differs from that. By definition we have

$$\begin{aligned}\Sigma^{(d)} &= \langle \frac{1}{N} AA^\dagger \rangle \\ &= \int \frac{1}{N} AA^\dagger \bar{w}(A|\Sigma, N) d[A] .\end{aligned}\quad (16)$$

Inserting Eq. (8), we can do the ensemble average in the Gaussian case which yields the covariance matrix $N\Sigma/\eta$. Thus, only the η integral remains and we have

$$\Sigma^{(d)} = N\Sigma \int_0^\infty \frac{f(\eta)}{\eta} d\eta = N\Sigma \overline{\eta^{-1}} \quad (17)$$

implying that the two covariance matrices differ by the average of $1/\eta$. Alternatively, one can calculate $\Sigma^{(d)}$ from the amplitude distribution,

$$\Sigma^{(d)} = \langle rr^\dagger \rangle = \int rr^\dagger \langle g \rangle(r|\Sigma, N) d[r] , \quad (18)$$

which yields

$$\Sigma^{(d)} = \Sigma \int_0^\infty xp(x) dx = \Sigma \bar{x} . \quad (19)$$

Here, the two covariance matrices differ by the first moment of x . With the help of Eq. (14), the results (17,19) are easily seen to coincide.

Having extracted the covariance matrix for the total time interval from the data, we can proceed with the determination of the deformation functions. The exponential function in the integrand of Eq. (15) allow us to interpret it as a Laplace transform,

$$\frac{\Gamma(N/2)}{2} \frac{p(x)}{x^{N/2-1}} = \mathcal{L} \left(\tilde{\eta}^{N/2} f(2\tilde{\eta}) \right) , \quad (20)$$

where we introduced $\tilde{\eta} = \eta/2$ to avoid inconvenient factors of two. Thus, the ensemble deformation function is the inverse Laplace transform

$$f(2\tilde{\eta}) = \frac{\Gamma(N/2)}{2} \frac{1}{\tilde{\eta}^{N/2}} \mathcal{L}^{-1} \left(\frac{p(x)}{x^{N/2-1}} \right) . \quad (21)$$

of the amplitude distribution deformation function divided by a power of x . This makes it possible to determine $f(\eta)$ by extracting $p(x)$ from the amplitude time series and carrying out the inverse Laplace transform. In contrast, extracting $f(\eta)$ directly from the data is cumbersome and burdened by limited statistics, as the following discussion shows. The rows of A are the model time series of length N and cannot easily be identified with the amplitude time series $r_k(t)$ of length T . However, the matrices AA^\dagger/N form the ensemble of model covariance matrices and can be compared with the empirical ones. As a certain sample length is required for

meaningful results, it is out of question to compare the matrices directly, *i.e.* their individual matrix elements. A better observable is the distribution

$$q(s) = \int \delta \left(s - \frac{1}{N} \text{tr} AA^\dagger \right) \bar{w}(A|\Sigma, N) d[A] \quad (22)$$

of the traces, which can easily be written as a single integral involving the ensemble deformation function $f(\eta)$. The distribution (22) is empirically obtained by moving a time window through the amplitude time series and calculating the empirical covariance matrices and their traces. This then gives $f(\eta)$.

The problem with the above procedure is its still limited statistical significance. Instead, extracting the amplitude distribution deformation function $p(x)$ from the data gives much more meaningful results. As we discuss in the sequel, the number of data points is larger by a factor of K . The amplitudes r_k appear in Eq. (13) only via the bilinear form $r^\dagger \Sigma r$. We rotate the amplitude vector r into the eigenbasis of the empirically obtained covariance matrix Σ . By definition, the eigenvalues of Σ are positive and larger than zero, provided the length of the sampling interval is larger than K . We divide each component of the rotated amplitude vector by the square root of the corresponding eigenvalue and denote the resulting vector by \tilde{r} . Within our model, all components of \tilde{r} should be equally distributed. We integrate out all but one, \tilde{r}_k , and arrive at the distribution

$$\langle \tilde{g} \rangle(\tilde{r}_k) = \int_0^\infty p(x) \frac{1}{\sqrt{2\pi x}} \exp \left(-\frac{\tilde{r}_k^2}{2x} \right) dx . \quad (23)$$

Thus, $p(x)$ may be identified with the distribution of the variances x of the Gaussian distributed random variables \tilde{r}_k . Conceptually, this is our main result. It provides a simple and statistically significant method to obtain the amplitude distribution deformation function $p(x)$ which then yields upon inverse Laplace transform the ensemble distribution function $f(\eta)$. As we have K time series $r_k(t)$, we gain a factor of K by aggregation.

III. APPLICATION TO FINANCIAL DATA

We now apply our method to stock market data. This is of particular interest as heavy tails are ubiquitous in finance. A better modeling for multivariate distributions is urgently called for to improve risk estimation. We present the data in Sec. III A, extract the deformation functions in Sec. III B and calculate the ensemble and return distributions in Sec. III C.

A. Data Set

We analyze the $K = 306$ continuously traded stocks with prices $S_k(t)$, $k = 1, \dots, K$ in the S&P 500[®] index between 1992 and 2012 [41], which we previously

analyzed with a purely Gaussian, *i.e.*, non-deformed Wishart ensemble [16]. The amplitudes are here the dimensionless price changes

$$R_k(t) = \frac{S_k(t + \Delta t) - S_k(t)}{S_k(t)}, \quad (24)$$

which are referred to as returns. They depend on the chosen return horizon Δt . According to Eq. (3), we calculate the returns $r_k(t)$ normalized to zero mean. To make our presentation self-contained, we show once more how strongly the whole $K \times K$ correlation matrix C for this data set changes in time. In Fig. 1, it is displayed for subsequent three-months time windows. In most ran-

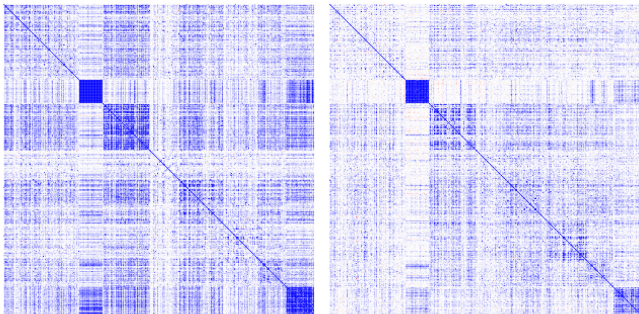


FIG. 1: Correlation matrices of $K = 306$ companies in S&P 500[®] for the fourth quarter of 2005 and the first quarter of 2006. The darker, the stronger the correlation. The companies are sorted according to industrial sectors. Taken from Ref. [16].

dom matrix approaches, the ensemble is fictitious and enters only by means of an ergodicity argument. This is not so here, as Fig. 1 illustrates. Our ensemble exists in reality, it is the whole set of matrices analyzed by moving a sample time window through the data. In Fig. 1, one also sees rather stable stripes in these correlation matrices which are due to the different industrial sectors, see, *e.g.*, Ref. [11]. Obviously, basis invariance is not present in this data set, and probably neither in any other real data set. Hence, a direct extraction of the ensemble deformation function $f(\eta)$ which preserves the basis invariance of the random matrix ensemble is problematic. Yet, there is still another reason: market states were identified which reveal a fine structure of the ensemble [11, 42]. As every random matrix ensemble has an effective character, one is advised to analyze quantities which already reflect this. In the present case, such quantities are the amplitude, in the present case return, distribution and the corresponding deformation function $p(x)$.

B. Deformation Functions

We use daily data, *i.e.*, $\Delta t = 1$ trading day. Rotation of the return vector r into the eigenbasis of the empirical

covariance matrix Σ , normalization to the square roots of the eigenvalues and aggregation on a five-day window yield the empirical distributions of variances shown in Fig. 2. Aggregation on a ten-day window gives similar

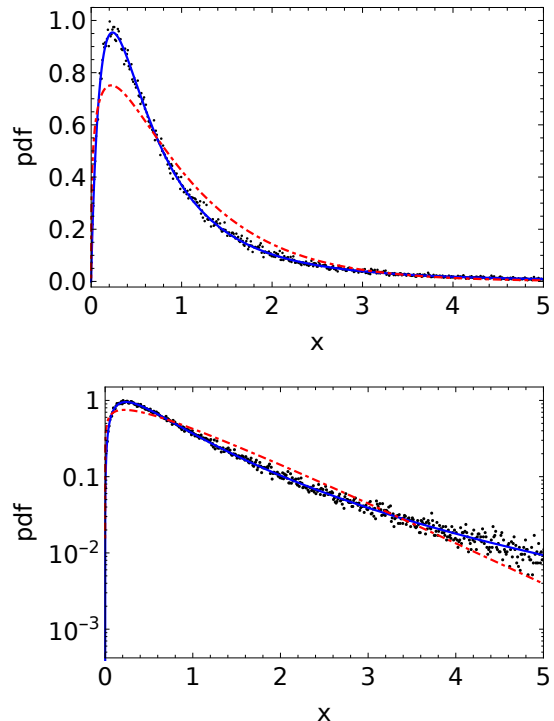


FIG. 2: Aggregated distribution of variances as dots, calculated on a five-day time window on a linear (top) and logarithmic (bottom) scale. Fit to a beta prime distribution as solid line. For comparison, a χ^2 distribution as dashed dotted line.

results; hence, the estimation noise does not have a major impact on the distribution. A variety of functions is capable of describing the data. In finance, one often employs the log-normal distributions to model volatilities, see *e.g.* Ref. [43]. In finance, the standard deviations are referred to as volatilities. However, the log-normal distribution fails to capture the empirically found tail behavior. More suitable is the beta prime distribution

$$p(x|N, L) = \frac{\Gamma(N + L/2)}{\Gamma(N/2)\Gamma((N + L)/2)} \frac{x^{N/2-1}}{(1 + x)^{N+L/2}} \quad (25)$$

with two positive parameters N and L . Anticipating the following discussion, we choose their combination in the expression (25) in such a way that N coincides with the parameter N introduced in Sec. II. The fit is depicted in Fig. 2, the agreement with the data is much better than for a χ^2 distribution corresponding to the ensemble of Ref. [16] which is formally obtained by setting $f(\eta) = \delta(\eta - 1)$ or $f(\eta) = \delta(\eta - N)$, respectively, depending on the rescaling with N . We carry out fits for different return horizons Δt . The results for N and L are shown in

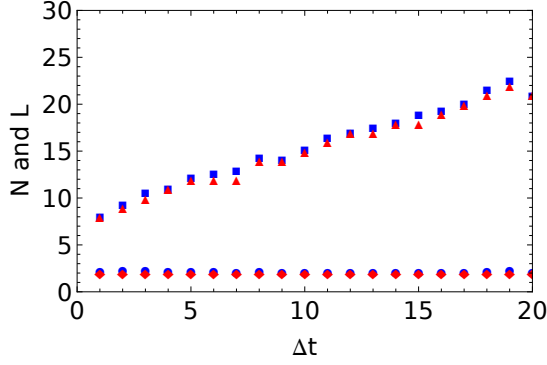


FIG. 3: Fitted values of N (upper, ascending dots) and L (lower dots) for the beta prime distribution versus the return horizon Δt . Triangles and diamonds for the fits with constraint to integer values, squares and circles without constraint.

Fig. 3. While L stays constant around two, N increases from about seven for daily data to about 23 for $\Delta t = 19$ trading days. We postpone the interpretation up to the evaluation of the ensemble and return distribution.

Having extracted the amplitude, *i.e.*, return distribution deformation function $p(x|N, L)$, we calculate the inverse Laplace transform (21),

$$f(2\tilde{\eta}|N, L) = \frac{\Gamma(N + L/2)}{2\tilde{\eta}^{N/2}\Gamma((N + L)/2)} \mathcal{L}^{-1} \left((1 + x)^{-N - L/2} \right), \quad (26)$$

and find [44] with $\eta = 2\tilde{\eta}$ for the ensemble deformation function

$$\begin{aligned} f(\eta|N, L) &= \frac{\eta^{(N+L)/2-1}}{2^{(N+L)/2}\Gamma((N + L)/2)} \exp\left(-\frac{\eta}{2}\right) \\ &= \chi_{N+L}^2(\eta). \end{aligned} \quad (27)$$

This is a χ_{N+L}^2 distribution with $N + L$ degrees of freedom. As required, $f(\eta|N, L)$ is a positive and normalized function.

C. Deformed Ensemble and Return Distribution

Inserting Eq. (27) into Eq. (8) yields after a straightforward calculation

$$\begin{aligned} \overline{w}(A|\Sigma, N, L) &= \frac{\Gamma((N + NK + L)/2)}{\Gamma((N + L)/2) \det^{N/2}(\pi N \Sigma)} \\ &\quad \left(1 + \frac{\text{tr } A^\dagger \Sigma^{-1} A}{N} \right)^{-(N + NK + L)/2} \end{aligned} \quad (28)$$

for the distribution of the random matrices A . Thus, we arrive at an ensemble characterized by an algebraic distribution. For a similar ensemble, but in the special case of $\Sigma = 1_K$, spectral correlation functions were studied

in Refs. [45, 46]. Here, however, we derived our ensemble from data, and the dependence on a non-trivial Σ is in the present essential. Anticipating the result (28), we rescaled Σ with N as compared to Ref. [16]. Thereby, N and L appear on equal footing in the formulae. To obtain the ensemble averaged return distribution, we plug Eq. (25) into Eq. (13) and find

$$\begin{aligned} \langle g \rangle(r|\Sigma, N, L) &= \frac{\Gamma(N + L/2)\Gamma((N + K + L)/2)}{\Gamma(N/2)\Gamma((N + L)/2)\sqrt{\det(\pi N \Sigma)}} \\ &\quad \mathcal{U}\left(\frac{N + K + L}{2}, \frac{K - N + 2}{2}, \frac{r^\dagger \Sigma^{-1} r}{2}\right) \end{aligned} \quad (29)$$

with the confluent hypergeometric function [47]

$$\mathcal{U}(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty y^{a-1} (1+y)^{b-a-1} \exp(-yz) dy \quad (30)$$

for positive real parts of a and z . From Eq. (17) or (19) the covariance matrix

$$\Sigma^{(d)} = \frac{N}{N + L - 2} \Sigma \quad (31)$$

for the deformed ensemble follows. To compare with the empirical return distribution, we compute the integral (23),

$$\begin{aligned} \langle \tilde{g} \rangle(\tilde{r}_k|N, L) &= \frac{\Gamma(N + L/2)\Gamma((N + L + 1)/2)}{\Gamma(N/2)\Gamma((N + L)/2)\sqrt{2\pi}} \\ &\quad \mathcal{U}\left(\frac{N + L + 1}{2}, \frac{3 - N}{2}, \frac{\tilde{r}_k^2}{2}\right). \end{aligned} \quad (32)$$

A comment on the permissible values for the parameter N is in order. In the return distributions (29) and (32), N can take any positive real value. In the ensemble distribution (28), however, N is the length of the model time series or, equivalently, one of the dimensions of the $K \times N$ matrices A and thus restricted to integer values. It is thus a matter of interpretation whether one wants to impose the constraint that N be integer. There is no such restriction for the parameter L . In any case, we also carried out fits with the integer constraint. The results shown in Fig. 3 do not indicate a strong influence of this constraint.

The results of the data comparison are displayed in Fig. 4 for daily returns, $\Delta t = 1$ trading day. The fitted parameter values are $N = 8.13$ and $L = 2.24$. The center of the empirical distribution is slightly better described by employing the deformed Wishart ensemble instead of the non-deformed one in Ref. [16]. The heavy tails clearly reveal that the deformed Wishart ensemble yields overall a better description, since the result of Ref. [16] consistently underestimates the large events. In Fig. 5 we present the same analysis for returns with $\Delta t = 20$ trading days, the fit gives $N = 20.98$ and $L = 2.07$. Here, the tails are still strong, but less pronounced than for daily data. For the interpretation of these results, we

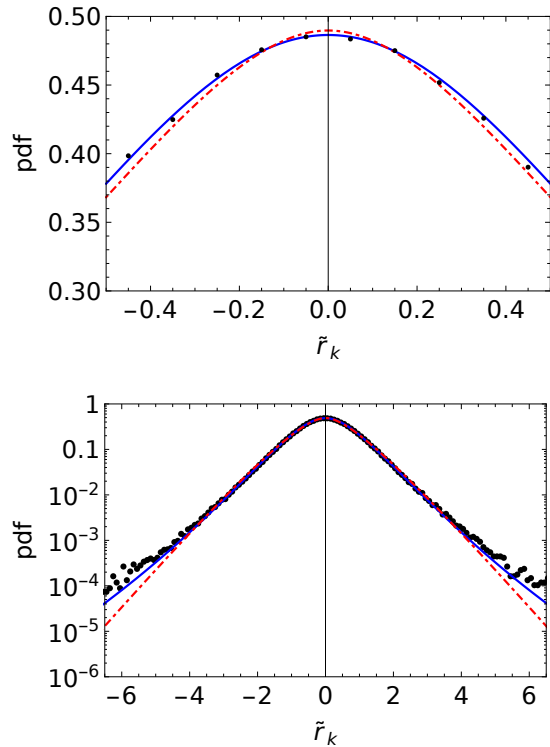


FIG. 4: Aggregated distribution of daily returns, $\Delta t = 1$ trading day. Empirical results as dots, fit to the distribution (32) as solid line. The corresponding result of Ref. [16] as dashed line. Center of the distribution on a linear scale (top), whole distribution on a logarithmic scale (bottom).

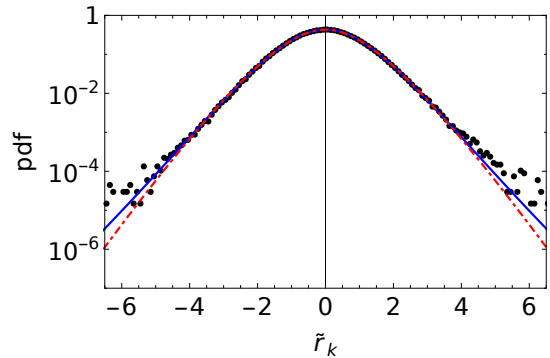


FIG. 5: Same as Fig. 4 (bottom) for returns with $\Delta t = 20$ trading days.

recall the well-established fact that *univariate* distributions of returns for one stock acquire heavy tails as the return horizon Δt becomes smaller, see *e.g.* Ref. [48]. Here, however, we analyze the *multivariate* distribution of K correlated stocks. Thus, there are two competing effects. First, as discussed in general in connection with Eq. (4) and for the financial data in Ref. [16], the su-

perposition of the amplitudes, in the present case the returns, drives the multivariate distribution towards a Gaussian, provided that the covariances are sufficiently constant. Second, as observed in Ref. [16] and extended here, the fluctuations of the non-stationary covariances lift the tails of the distributions evaluated over long time intervals and make them heavier. Not surprisingly, the heavier the tails of the *univariate* distributions, the heavier are also those of the ensemble averaged multivariate ones shown above. This is nicely reflected in the nearly linear increase of the parameter N on the return horizon Δt , see Fig. 3. The smaller N , the heavier are the tails in the ensemble distributions (28) and in the ensemble averaged return distribution (29). As N grows and L is held fixed, the distribution (28) comes closer to a Gaussian, *i.e.*, to the non-deformed ensemble. It is quite remarkable that the fit of L always yields values close to two. According to Eq. (31), this implies $\Sigma^{(d)} \approx \Sigma$. Put differently, the heavy tails in the cases considered alter the measured covariances only slightly as compared to a Gaussian assumption. When looking at financial risk, however, the tails are very important.

Finally, we use the opportunity to discuss an issue of general interest when presenting and fitting a multivariate distribution that depends on the statistical variables only via a bilinear form such as $r^\dagger \Sigma^{-1} r$. Instead of the above procedure which involves rotation of r into the eigenbasis of Σ and aggregation, one might also view the bilinear form as a generalized radius

$$\rho = \sqrt{r^\dagger \Sigma^{-1} r} \quad (33)$$

in the K dimensional space and study its distribution

$$\langle g_{\text{rad}} \rangle(\rho) = \int \delta(\rho - \sqrt{r^\dagger \Sigma^{-1} r}) \langle g \rangle(r | \Sigma, N, L) d[r] . \quad (34)$$

A rather straightforward calculation yields

$$\langle g_{\text{rad}} \rangle(\rho) = \frac{\Gamma(N + L/2) \Gamma((N + K + L)/2)}{2^{K/2-1} \Gamma(N/2) \Gamma(K/2) \Gamma((N + L)/2)} \rho^{K-1} \mathcal{U} \left(\frac{N + K + L}{2}, \frac{K - N + 2}{2}, \frac{\rho^2}{2} \right) . \quad (35)$$

The Jacobian ρ^{K-1} , typical for such a radial distribution, appears and, because of $K = 306$, dominates the functional form of the distribution for small ρ , as can be seen in Fig. 6. The theoretical result (35) describes the data much better than the corresponding result of Ref. [16]. Nevertheless, the dominance of ρ^{K-1} gives a somewhat misleading picture and we infer that using the distributions (29) and (32) is more appropriate.

IV. PERMISSIBILITY OF DEFORMATION FUNCTIONS

When extracting the return distribution deformation function $p(x)$ from the data, we encountered a puzzling

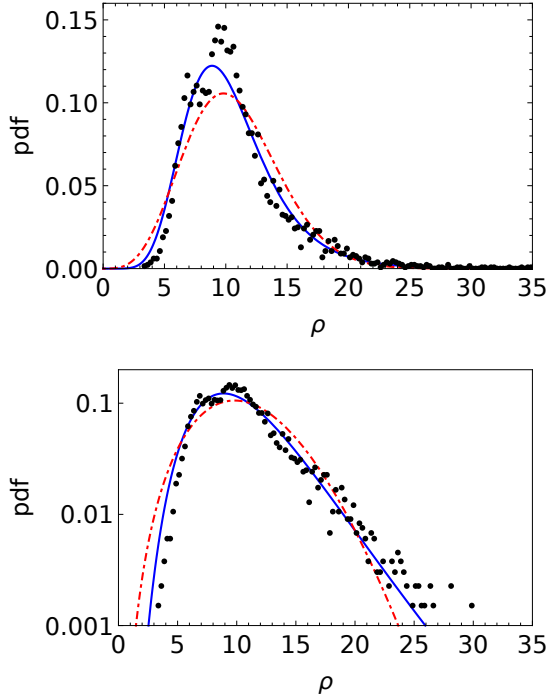


FIG. 6: Radial distribution of daily returns, $\Delta t = 1$ trading day. Empirical results as dots, fit to the distribution (35) as solid line. The corresponding result of Ref. [16] as dashed line, on a linear (top) and a logarithmic scale (bottom).

problem that we wish to report here. The log-logistic distribution

$$p(x|b, c) = \frac{b}{c} \frac{(x/c)^{b-1}}{(1 + (x/c)^b)^2} \quad (36)$$

with $b = N/2$ yields a very good description of the data, even slightly better than the beta prime distribution. The c values are around one, $N = 4$ for $\Delta t = 1$ trading day and increasing for larger Δt . However, the resulting ensemble deformation function can take positive and negative values. For example, for $N = 4$, we find

$$f(\eta|c) \sim \frac{\sin(\eta/2) - \eta \cos(\eta/2)/2}{\eta^2}. \quad (37)$$

Hence, it cannot be interpreted as a distribution. Indeed, we are confronted with a problem of interpretation. While the discussion in Sec. II C clearly revealed that $p(x)$ is a well-defined distribution of a random variable, namely of a variance, it is not obvious that $f(\eta)$ also represents a well-defined distribution. The corresponding random variables, the matrices A , do not have a direct data interpretation. Thus, one might simply view $f(\eta)$ as a continuous coefficient function for the expansion of the distribution (8) in terms of Gaussians. Nevertheless, even if one does not want to enforce an interpretation of $f(\eta)$ as a distribution, the resulting ensemble distribution (8)

must be positive definite. We test this by calculating the distribution of the traces

$$u(s) = \int \delta \left(s - \frac{1}{N} \text{tr} A^\dagger \Sigma^{-1} A \right) \bar{w}(A|N, c) d[A]. \quad (38)$$

As we only wish to test the positivity, it is convenient to choose it different from the above distribution (22) by including the covariance matrix. After some algebra, we can express it as a high-order derivative involving the return distribution deformation function,

$$u(s) = \frac{(-1)^{(K-1)N/2} \Gamma(N/2)}{\Gamma(KN/2)} s^{KN/2-1} \frac{d^{(K-1)N/2}}{ds^{(K-1)N/2}} \frac{p(s|b, c)}{s^{N/2-1}}. \quad (39)$$

This in principle general result gives for the log-logistic distribution (36)

$$u(s) = \frac{(-1)^{(K-1)N/2} N c^{N/2} \Gamma(N/2)}{2\Gamma(KN/2)} s^{KN/2-1} \frac{d^{(K-1)N/2}}{ds^{(K-1)N/2}} \frac{1}{(c^{N/2} + s^{N/2})^2}. \quad (40)$$

Restricting ourselves to even N , we may employ the theory of complex functions to calculate the pole expansion

$$\frac{1}{(c^{N/2} + s^{N/2})^2} = \sum_{n=1}^{N/2} \frac{1}{\prod_{m \neq n} (a_n - a_m)^2} \left(\frac{1}{(s - a_n)^2} - \frac{2}{s - a_n} \sum_{l \neq n} \frac{1}{a_n - a_l} \right) \quad (41)$$

with the poles

$$a_n = c \exp \left(\frac{i2\pi}{N} (2n+1) \right). \quad (42)$$

The derivatives in Eq. (40) can now easily be evaluated and we arrive at

$$u(s) = \frac{N c^{N/2} \Gamma(N/2)}{2\Gamma(KN/2)} s^{KN/2-1} \sum_{n=1}^{N/2} \frac{1}{\prod_{m \neq n} (a_n - a_m)^2} \left(\frac{\Gamma((K-1)N/2 + 2)}{(s - a_n)^{(K-1)N/2+2}} - \frac{2\Gamma((K-1)N/2 + 1)}{(s - a_n)^{(K-1)N/2+1}} \sum_{l \neq n} \frac{1}{a_n - a_l} \right). \quad (43)$$

Inspite of the complex poles, this is by construction a real function. Yet, it takes positive and negative values which outrules an interpretation of $u(s)$ and thus also of $\bar{w}(A|N, c)$ as distributions. By means of this example we face the somewhat surprising result that a well-defined distribution $p(x)$ does not necessarily yield a well-defined ensemble. Each case has to be investigated individually.

V. FURTHER EXTENSION BY DEFORMING THE STATIC AMPLITUDE DISTRIBUTION

We argued in Sec. IIA that the Gaussian assumption (4) for the static amplitude distribution is not as restrictive as it might appear at first sight. Nevertheless, we now extend our construction by assuming more general functional forms. At present, we do not have data at our disposal in which the static amplitude distribution is non-Gaussian, but we nevertheless now extend our construction, as it might be useful for future data analyses. Moreover, we will also come across some interesting observations. Instead of Eq. (4), we now assume that the static amplitude distribution can be expressed as an average over the Gaussian (4),

$$\bar{g}(r|\Sigma_s) = \int_0^\infty h(\xi) g\left(r \left| \frac{\Sigma_s}{\xi} \right.\right) d\xi \quad (44)$$

with a new deformation function $h(\xi)$ that fulfils

$$\int_0^\infty h(\xi) d\xi = 1 \quad \text{and} \quad h(\xi) \geq 0. \quad (45)$$

We proceed as in Sec. IIB. Instead of the ensemble average (6), we now have

$$\langle \bar{g} \rangle(r|\Sigma, N) = \int d[A] \bar{w}(A|\Sigma, N) \bar{g}\left(r \left| \frac{1}{N} A A^\dagger \right.\right). \quad (46)$$

This is readily cast into the form

$$\langle \bar{g} \rangle(r|\Sigma, N) = \int_0^\infty p(x) g(r|x\Sigma) dx, \quad (47)$$

which differs from Eq. (13) only by the definition of the amplitude distribution deformation function. It is now given by

$$p(x) = \int_0^\infty d\xi h(\xi) \int_0^\infty d\eta f(\eta) \int_0^\infty dz \chi_N^2(z) \delta\left(x - \frac{z}{\xi\eta}\right). \quad (48)$$

For fixed ξ , we introduce the new variable $\hat{\eta} = \eta\xi$ and find

$$p(x) = \int_0^\infty d\hat{\eta} \hat{f}(\hat{\eta}) \int_0^\infty dz \chi_N^2(z) \delta\left(x - \frac{z}{\hat{\eta}}\right). \quad (49)$$

which coincides with Eq. (14), but now involving the new ensemble deformation function

$$\hat{f}(\hat{\eta}) = \int_0^\infty \frac{h(\xi)}{\xi} f\left(\frac{\hat{\eta}}{\xi}\right) d\xi. \quad (50)$$

This integral is reminiscent of a convolution. Thus, the case of a deformed, non-Gaussian static amplitude distribution is formally traced back to the Gaussian case. The difference can be fully absorbed into the ensemble deformation function. Importantly, this means that all other results of Sec. II continue to hold, in particular the Laplace transform (20) and its inversion (21). Nevertheless, the following problem remains. We can extract $h(\xi)$ and $p(x)$ from the data by using the methods outlined in Sec. IIC for very short time intervals and for the whole, long time interval, respectively. From the inverse Laplace transform (21), we obtain $\hat{f}(\hat{\eta})$, but to determine $f(\eta)$, we are left with the task to invert Eq. (50). Although that is definitely possible for some special cases, a general inversion formula is lacking. In practical applications, however, the extension sketched above is more likely to be needed for consistency tests. For example, if some of the available data for the same system permit the Gaussian assumption for the static amplitude distribution and others do not, one can first determine $f(\eta)$ as described in Sec. II and then turn to the data which require an additional deformation function $h(\xi)$. Once both of these deformation functions are known, one can evaluate $\hat{f}(\hat{\eta})$ and check if it is consistent with the inverse (21) of $p(x)$ which is independently extracted from the data.

As an example, we consider the case that both, $f(\eta)$ and $h(\xi)$, are χ^2 distributions

$$f(\eta) = \chi_{N+L}^2(\eta) \quad \text{and} \quad h(\xi) = \chi_M^2(\xi) \quad (51)$$

of $N+L$ and M degrees of freedom, respectively. The choice for $f(\eta)$ coincides with the result of Sec. IIIB. With Eq. (50), we obtain

$$\hat{f}(\hat{\eta}) = \frac{\sqrt{\hat{\eta}}^{(N+L+M)/2-2} \mathcal{K}_{(N+L-M)/2}(\sqrt{\hat{\eta}})}{2^{(N+L+M)/2-1} \Gamma((N+L)/2) \Gamma(M/2)}, \quad (52)$$

where \mathcal{K}_ν is the modified Bessel function of the second kind of order ν . This function already appeared in the ensemble averaged return distribution of Ref. [16]. According to Eq. (49), the distribution $p(x)$ is an integral involving the modified Bessel function and the return distribution averaged over the deformed ensemble is an integral over a product of modified Bessel functions, but we do not give the formulae here.

VI. CONCLUSIONS

Non-stationarity is an often encountered feature in complex systems. Here, we addressed non-stationarity of correlations. We presented a method to determine their distribution from the amplitude distribution. Put differently, we showed how to extract the proper ensemble of random covariance matrices from amplitude data. Carrying out our analysis for the case of financial data, we found an algebraic distribution of covariance matrices reflecting the heavy tails in the amplitude, *i.e.*, return distributions.

A conceptually important comment is in order. Consider any two empirical covariance matrices $\Sigma(t_1)$ and $\Sigma(t_2)$. They are certainly not independent, because, first, the correlation structure due to, *e.g.*, the industrial sectors in the financial markets only changes on a large time scale and, second, one would expect that they are the more dependent the smaller the time difference $|t_1 - t_2|$. The first cause is a built-in feature of our model, as the random model covariance matrices fluctuate around the empirical Σ . The second cause is effectively accounted for as the length of our model time series N , *i.e.*, the second dimension of the random matrices A , is different from the length T of the subintervals in which the empirical covariance matrices $\Sigma(t)$ are evaluated. The values for N resulting from the data analysis are very small $N \ll K$ while $T \geq K$ when measuring $\Sigma(t)$. Our random matrix ansatz does not aim at modeling the ensemble of the empirical covariance matrices $\Sigma(t)$ in a one-to-one fashion. This is never the goal of a statistical approach. Our model has an effective character which is obvious, for example, in the relation $N \ll T$. Model time series much shorter than the empirical ones suffice to properly grasp the statistical effects induced by the fluctuations around the average covariance matrix. This also reflects the mutual dependence of the empirical covariance matrices. We demonstrated by our analysis of financial data how useful our model is.

Furthermore, observables such as the amplitude distribution do not resolve autocorrelations in time of any kind. Even reshuffling the order of the empirical covariance matrices $\Sigma(t)$ in time does not change the amplitude distribution for the total time interval. A similar situation is encountered when analyzing statistical properties of quantum chromodynamics. The gauge fields may be viewed as a really existing ensemble over which an average is carried out — this is the very definition of the partition function. Although these gauge fields are not independent either, an effect of this autocorrelation is only seen if corresponding observables are used. Densities are not affected. Random matrices as models for the highly non-trivial gauge fields are applied with great success.

Yet another aspect is worth mentioning. In contrast to random matrix applications for Hamiltonian systems,

there is not a local scale that can enforce universal statistical behavior. Thus, it is important how the covariances are actually distributed. Gaussian assumptions are only acceptable if really justified by data analysis. The present study extends a previous one in which we employed such a Gaussian assumption in finance. Here, we reconsidered the same data set and clearly demonstrated that the Gaussian assumption underestimates the tails. The algebraic distributions that we found here are relevant for risk estimation as they help to better understand large events. Importantly, once the ensemble is properly extracted, meaningful averages can be computed for all observables that depend on non-stationary covariances.

When developing our construction, we came across a puzzling feature which calls for a caveat. The deformation function extracted from the amplitude distribution determines, on the one hand, uniquely the ensemble, but, on the other hand, this ensemble is not necessarily well-defined. Each case has to be studied individually. We do not expect this to cause severe problems in applications, but conceptually it is an interesting aspect. We also extended our construction by including deformed static amplitude distributions. The additional freedom accompanying this extension might offer a possibility to circumvent the above mentioned puzzling problem. From a more general viewpoint, we have to emphasize that our construction only includes functional forms of the ensemble that depend on the trace over the product of the random covariance matrix and the mean covariance matrix. Although this is quite natural, as it guarantees a certain amount of basis invariance which all random matrix models need, more general functional forms pose an interesting and potentially important challenge.

Hitherto, we only applied our method to finance. We plan applications to other complex systems, too. This may be rewarding, as large events and risk estimations are not only important in finance.

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